Quadratic Equations

QUADRATIC EQUATIONS

SERIES K

EVERYTHING YOU NEED TO KNOW
100%
AND NOTHING MORE

MATHLETICS

www.mathletics.com.au

QUADRATIC EQUATIONS

A quadratic equation is an equation where the highest index is 2 (squared). These are a little more complicated to solve than linear equations, and may have no real solutions at all.

Answer these questions, *before* working through the chapter.

I used to think:

How many solutions does a quadratic equation have?

Is it possible to find solutions for x in $ax^2 + bx + c = 0$ based only on a, b and c?

If $x^2 = 16$, is $x = 4$ the only answer?

Answer these questions, *after* working through the chapter.

But now I think:

How many solutions does a quadratic equation have?

Is it possible to find solutions for x in $ax^2 + bx + c = 0$ based only on a, b and c?

If $x^2 = 16$, is $x = 4$ the only answer?

What do I know now that I didn't know before?

Quadratic Equations

If the highest index of the variable is 2 (eg. x^2 or y^2) then the equation is not linear, but is called a **quadratic equation**. To find the variable by itself on one side, the square root is used to change x^2 to x. But the **square root must be used on both sides.**

Solve for x in the quadratic equation

$$2x^2 + 16 = 34$$

$$2x^2 + 16 - 16 = 34 - 16 \quad \leftarrow \text{Subtract 16 from both sides}$$

$$\frac{2x^2}{2} = \frac{18}{2} \quad \leftarrow \text{Divide both sides by 2}$$

$$x^2 = 9$$

$$\text{Find the square root of both sides} \rightarrow x^2 = (\pm 3)^2 \quad \leftarrow \text{Because } 3^2 = 9 \text{ and } (-3)^2 = 9$$

$$x = \pm 3$$

Each quadratic equation can have one, two or no real solutions. The above example has two solutions ($x = -3$ or $x = 3$). Sometimes the solutions will have to be left in surd form like this next example:

Solve for x

$$4x^2 - 20 = 0$$

$$4x^2 = 20$$

$$x^2 = 5$$

$$x^2 = (\pm\sqrt{5})^2$$

$$x = \pm\sqrt{5}$$

This equation has the two possible solutions $x = \sqrt{5}$ or $x = -\sqrt{5}$

Here is an example of a quadratic equation with no real solutions:

Solve for x in the quadratic equation

$$3x^2 + 8 = -4$$

$$3x^2 + 8 - 8 = -4 - 8$$

$$\frac{3x^2}{3} = \frac{-12}{3}$$

$$x^2 = -4$$

$$x = \pm\sqrt{-4}$$

This does not have any real solutions since no real number is the square root of a negative number.

1. What is the difference between a linear and a quadratic equation?

2. Find the solutions to the variable in each of these equations. Leave the solutions in surd form if necessary, and state if each equation has one, two or no real solutions:

a $x^2 = 4$

b $2b^2 = 32$

c $3b^2 - 75 = 0$

d $6y^2 = 0$

e $4t^2 - 28 = 0$

f $h^2 = -9$

g $2p^2 = -8$

h $8m^2 + 5 = 5$

i $9x^2 - 16 = 0$

j $-3k^2 + 108 = 0$

The Null Factor Law

Let's say there are two numbers a and b.

If $a \times b = 0$ then $a = 0$ or $b = 0$ (or both are 0)

This means that if the product of two expressions is zero, then at least one of the expressions must be equal to zero. Here are some examples of how the null factor law is used to solve equations:

Solve for x

a $x(x+3) = 0$

$\therefore x \times (x+3) = 0$

$\therefore x = 0$ or $x + 3 = 0$ (null factor law)

$\therefore x = 0$ or $x = -3$

b $(x-1)(x-2) = 0$

$\therefore (x-1) \times (x-2) = 0$

$\therefore x - 1 = 0$ or $x - 2 = 0$ (null factor law)

$\therefore x = 1$ or $x = 2$

Factorising to Solve Quadratic Equations

Type 1: $ax^2 + bx = 0$ → Factorise to 1 bracket and then use the null factor law to find two possible solutions.

Solve for x in these quadratic equations

a $x^2 + 4x = 0$

$x^2 + 4x = 0$

$x(x+4) = 0$

$x \times (x+4) = 0$

$\therefore x = 0$ or $x + 4 = 0$

$\therefore x = 0$ or $x = -4$

b $3x^2 - 15x = 0$

$3x^2 - 15x = 0$

$x(3x - 15) = 0$

$x \times (3x - 15) = 0$

$\therefore x = 0$ or $3x - 15 = 0$

$\therefore x = 0$ or $x = 5$

Type 2: $ax^2 + bx + c = 0$ → Factorise to 2 brackets and then use the *null factor law* to find two possible solutions.

Solve for x in these quadratic equations

a $x^2 + x = 12$

$x^2 + x - 12 = 0$

$\therefore (x-3)(x+4) = 0$

$\therefore x - 3 = 0$ or $x + 4 = 0$

$\therefore x = 3$ or $x = -4$

b $3x^2 + 14x - 5 = 0$

$\therefore (3x-1)(x+5) = 0$

$\therefore 3x - 1 = 0$ or $x + 5 = 0$

$\therefore 3x = 1$ or $x = -5$

$\therefore x = \frac{1}{3}$ or $x = -5$

1. Rewrite these polynomials in the form $ax^2 + bx + c = 0$. Identify the values of a, b and c:

a $x^2 + 3x + 2 = 0$

b $2x^2 + 4x + 5 = 0$

c $x^2 - 7 = 0$

d $x(x + 4) = 0$

e $3x(4x - 5) = 0$

f $(x + 3)(x - 7) = 0$

g $(3x + 5)(x - 8) = 0$

h $-3(x + 4)(x - 1) = 0$

2. Solve these quadratic equations for the missing variable:

a $x(x - 2) = 0$

b $x(x + 4) = 0$

c $x(2x + 1) = 0$

d $(x - 3)(x + 5) = 0$

e $y(2y - 3) = 0$

f $(2x + 7)(3x - 8) = 0$

3. Solve these quadratic equations by factorising:

a $x^2 + 8x = 0$

b $4x^2 + 12x = 0$

c $t^2 - 6t = 0$

d $5b^2 - 15b = 0$

e $y^2 - 4y - 21 = 0$

f $m^2 - m - 30 = 0$

g $2n^2 + 8n - 64 = 0$

h $x^2 + 5x = 6$

i $3x(x + 2) = 105$

j $(p + 3)^2 = 8p + 72$

Completing the Square

The trinomial $x^2 + 6x + 1$ can't be factorised into two brackets as easily as the examples in the previous section. So to solve $x^2 + 6x + 1 = 0$ the trinomial on the left needs to be transformed a different way. This is called **Completing the Square.**

$x^2 + 6x + 1$ is used in an example to show how to complete the square below:

Complete the square of $x^2 + 6x + 1$

Step 1: Write a bracket squared with x plus **half of the coefficient** of x

$$= x^2 + 6x + 1$$

Half

$$= (x + 3)^2$$

Subtract

Step 2: Subtract the square of the number in the bracket

$$= (x + 3)^2 - 9$$

Squared

Step 3: Write in the constant term from the trinomial

$$= (x + 3)^2 - 9 + 1$$

$$= (x + 3)^2 - 8$$

Step 4: Simplify

$$\therefore x^2 + 6x + 1 = (x + 3)^2 - 8$$

In the above example the coefficient of x^2 is 1. If it is not 1 then it must be factored out first.

Complete the square of $4x^2 - 16x - 8$

Step 1: Factor out the coefficient of x^2 using square brackets

$$= 4x^2 - 16x - 8$$

$$= 4[x^2 - 4x - 2]$$

Half

Step 2: Inside the square bracket – write a round bracket squared with x plus **half** the number in front of x in the square bracket

$$= 4[(x - 2)^2]$$

Subtract

Step 3: Inside the square bracket – subtract the square of the number in the round bracket

$$= 4[(x - 2)^2 - 4]$$

Squared

Step 4: Write the constant term in the square bracket from *Step 1*

$$= 4[(x - 2)^2 - 4 - 2]$$

$$= 4[(x - 2)^2 - 6]$$

$$= 4(x - 2)^2 - 24$$

$$\therefore 4x^2 - 16x - 8 = 4(x - 2)^2 - 24$$

4. Complete the square of the following trinomials:

a) $x^2 + 4x + 1$

b) $x^2 - 6x + 16$

c) $x^2 + 3x + 7$

Hint: $\left(\frac{3}{2}\right)^2 = \frac{9}{4}$

d) $3x^2 + 24x + 30$

e) $-x^2 + 10x - 23$

Hint: factorise -1 out using square brackets

f) $-3x^2 - 6x + 18$

Why is Completing the Square Important?

If a trinomial in an equation can't be factorised easily into two brackets, then completing the square can be used to solve the equation.

Solve for x if $x^2 - 6x + 1 = 0$

$x^2 - 6x + 1$ can't be factorised easily into two brackets. Completing the square can be used to show:

$$x^2 - 6x + 1 = (x)^2 - 8$$

So the original equation can be rewritten as: $(x - 3)^2 - 8 = 0$

Simplify so that the squared backet is by itself on the left hand side: $(x - 3)^2 - 8 + 8 = 0 + 8$

$$(x - 3)^2 = 8$$

Solve for the bracket by finding the square root of both sides: $(x - 3)^2 = (\pm\sqrt{8})^2$

$$\therefore x - 3 = \pm\sqrt{8}$$

Solve for the two possible solutions of x: $x - 3 = \sqrt{8}$ or $x - 3 = -\sqrt{8}$

$$\therefore x = 3 + \sqrt{8} \text{ or } x = 3 - \sqrt{8}$$

Solutions are usually left in surd form, unless a question specifies otherwise.

Solve the equation: $4x^2 - 16x - 8 = 0$

Using completing the square: $4x^2 - 16x - 8 = 4(x - 2)^2 - 24$

Rewrite the equation in the completing the square form: $4(x - 2)^2 - 24 = 0$

Simplify so that the squared bracket is by itself on the left hand side: $4(x - 2)^2 = 24$

$$(x - 2)^2 = 6$$

Solve for the bracket by finding the square root of both sides: $(x - 2)^2 = (\pm\sqrt{6})^2$

$$\therefore x - 2 = \pm\sqrt{6}$$

Solve for the two possible solutions of x: $x - 2 = \sqrt{6}$ or $x - 2 = -\sqrt{6}$

$$\therefore x = 2 + \sqrt{6} \text{ or } x = 2 - \sqrt{6}$$

5. Solve for x in the following:

a) $(x+2)^2 = 5$

b) $(x-3)^2 = 16$

c) $3(x-7)^2 = 8$

d) $-5(x+6)^2 = 35$

6. Complete the square and then solve for x in the following:

a) $x^2 + 6x + 2 = 0$

b) $x^2 - 10x + 15 = 0$

7. Complete the square and solve for the variable in the following:

a) $2q^2 + 3q - 2 = 0$

b) $-m^2 - 2m + 5 = 0$

c) $4t^2 + 8t - 1 = 0$

d) $-3x^2 + 12x - 2 = 0$

8. Consider the equation $ax^2+bx+c=0$ where a, b and c are any constants:

a Complete the square for this equation.

b Solve for x.

The Quadratic Formula

'The Quadratic Formula' is also know as 'The Formula' for quadratic equations. It is a formula that solves for any quadratic equation of the form $ax^2 + bx + c = 0$ by using only the coefficients a, b and c.

The formula is found from solving $ax^2 + bx + c = 0$ using completing the square.

$$ax^2 + bx + c = 0$$

$$\therefore x^2 + \frac{b}{a}x + \frac{c}{a} = 0$$

$$\therefore \left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a^2} + \frac{c}{a} = 0$$

$$\therefore \left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}$$

$$\therefore x + \frac{b}{2a} = \frac{\pm\sqrt{b^2 - 4ac}}{2a}$$

$$\therefore x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$\therefore x = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \quad \text{or} \quad x = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

This means that a solution can be found for a quadratic equation just using the coefficients.

Solve the following equations

$$x^2 + 5x + 3 = 0$$

$$a = 1, b = 5, c = 3$$

Quadratic formula → $$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Substitute a, b and c into quadratic formula → $$x = \frac{-5 \pm \sqrt{5^2 - 4(1)(3)}}{2(1)}$$

$$x = \frac{-5 \pm \sqrt{13}}{2}$$

$$x = -\frac{5}{2} + \frac{\sqrt{13}}{2} \quad \text{or} \quad x = -\frac{5}{2} - \frac{\sqrt{13}}{2}$$

Solutions are left in surd form unless a question says otherwise.

1. Use the quadratic formula to solve the following quadratic equations, leaving your answers in surd form:

a $x^2 - 3x - 18 = 0$

b $x^2 - 9x + 14 = 0$

c $2x^2 + 26x + 80 = 0$

d $-4t^2 + 32t - 60 = 0$

e $y^2 + 2y - 5 = 0$

f $-p^2 - 5p + 5 = 0$

2. Use the quadratic formula to solve the following quadratic equations, leaving your answers in surd form:

a $-3x^2 - 10x + 8 = 0$

b $4x^2 - 9 = 0$

c $x(2x - 7) = -4$

d $x(4 - 5x) = -6$

3. Solve $7x^2 + 42x - 112 = 0$ using each of the following methods.
(You should get the same solutions each time)

a) Factorise and solve for x.

b) Use completing the square to solve for x.

c) Use the quadratic formula to solve for x.

The Discriminant

Thanks to the quadratic formula we know that solutions to a quadratic equation $ax^2 + bx + c = 0$ can be found using:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Notice that if $b^2 - 4ac$ is negative then the surd can't be simplified - since there is no real number which is the square root of a negative number. So, if $b^2 - 4ac < 0$ then there are no real solutions for x.

The expression $b^2 - 4ac$ (the value under the surd in the quadratic formula) is called **the discriminant**, and represented by Δ ('Delta'). The discriminant predicts the properties of solutions to quadratic equations:

- If $\Delta = b^2 - 4ac < 0$ (**negative**) then there will be **no real solutions** for x.
- If $\Delta = b^2 - 4ac = 0$ then the two solutions will be **equal**. These equal solutions will also be rational.
- If $\Delta = b^2 - 4ac > 0$ (positive) and is a **perfect square** then there are **two solutions** which are **unequal and rational**.
- If $\Delta = b^2 - 4ac > 0$ (positive) and is **not** a perfect square, then there are **two solutions** which are **unequal and irrational**.

Have a look at these examples

a

$2x^2 + x - 7 = 0$

$\Delta = b^2 - 4ac = (1)^2 - 4(2)(-7) = 57$

$\therefore \Delta > 0$ and is not a perfect square

$\therefore$ the solutions will be unequal and irrational

Using the quadratic formula: $x = \dfrac{-1 \pm \sqrt{57}}{4}$

$\therefore x = \dfrac{-1 + \sqrt{57}}{4}$ or $x = \dfrac{-1 - \sqrt{57}}{4}$

Solutions are unequal and irrational

(As expected since $\Delta > 0$ and not a perfect square)

b

$x^2 + 10x + 25 = 0$

$\Delta = b^2 - 4ac = (10)^2 - 4(1)(25) = 0$

$\therefore \Delta = 0$

$\therefore$ the solutions will be equal

Using the quadratic formula: $x = \dfrac{-10 \pm \sqrt{0}}{2}$

$\therefore x = -5 + 0$ or $x = -5 - 0$

$\therefore \boldsymbol{x = -5}$ **solutions are equal**

Only one solution (as expected, since $\Delta = 0$)

c

$2x^2 + 3x + 3 = 0$

$\Delta = b^2 - 4ac = (3)^2 - 4(2)(3) = -15$

$\therefore \Delta < 0$

$\therefore$ there are no real solutions for x.

Using the quadratic formula: $x = \dfrac{-3 \pm \sqrt{-15}}{4}$

$\therefore$ the surd can't be simplified

$\therefore$ no real solutions for x

(As expected, since $\Delta < 0$)

The discriminant can also be used to determine coefficients.

Find values of k so that $3x^2 - 6x + k = 0$ has:

a Equal solutions

For solutions $\Delta = b^2 - 4ac = 0$

$\therefore (-6)^2 - 4(3)(k) = 0$

$\therefore 36 - 12k = 0$

$\therefore 12k = 36$

$\therefore k = 3$

$\therefore$ if $k = 3$ then the equation will have equal solutions

b No real solution

For no real solutions $\Delta = b^2 - 4ac < 0$

$\therefore (-6)^2 - 4(3)(k) < 0$

$\therefore 36 - 12k < 0$

$\therefore 12k > 36$

$\therefore k > 3$

$\therefore$ if k is greater than 3 then the equation will have no real solutions

c Two real and unequal solutions

For real and unequal solutions $\Delta = b^2 - 4ac > 0$

$\therefore (-6)^2 - 4(3)(k) > 0$

$\therefore 36 - 12k > 0$

$\therefore 12k < 36$

$\therefore k < 3$

$\therefore$ if k is less than 3 then the equation will have real and unequal solutions

$mx^2 - 5x = -6$ has equal solutions. Find m

First, write the equation in standard form: $mx^2 - 5x + 6 = 0$

$\therefore a = m, b = -5$ and $c = 6$

For equal roots

$\Delta = b^2 - 4ac = 0$

$\therefore (-5)^2 - 4(m)(6) = 0$

$25 - 24m = 0$

$\therefore m = \frac{25}{24}$

4. Use the discriminant to determine if these equations have two equal, unequal or no real solutions.
(Hint: Make sure the equation is in standard form $ax^2 + bx + c = 0$)

a $4x^2 + 21x - 18 = 0$

b $3x^2 + 7x + 2 = 0$

c $x^2 - 3x + 11 = 0$

d $2x^2 - 18 = 16x$

e $x = \frac{3}{x} - 4$

f $8x - 8 = 2x^2$

5. Find the value of r if $rx^2 - 3x + 9 = 0$:

a) Has one solution.

b) Has two unequal solutions.

6. Find the values for k such that $x^2 + x + 7k = 6k - 3x$ has no real solutions:

7. Use the discriminant to prove that $rx^2 + 2rx + r = 0$ has equal solutions for any r:

8. Prove that $2x^2 + 2kx + 2k^2 = 3kx - 3k^2$ never has real roots for any k:

9. Find p if the following equation has two unequal real solutions:

$$\frac{2}{p} = \frac{2 - 2x^2}{5 - 4x}$$

Equations that Reduce to Quadratics

Some equations can look very complicated, but they can be made easier by making a variable substitution to transform a nonquadratic equation into a quadratic equation.

This is done in 3 easy steps:

Step 1: Make a substitution to reduce the complicated equation to a quadratic equation

Step 2: Solve the quadratic equation

Step 3: Substitute the original variable back and solve

For example, solve for x if $2(2x-7)^2+3(2x-7)-20=0$

Step 1: substitute $k=2x-7$ to transform the complicated equation to

$$2k^2+3k-20=0$$

Step 2: this is a quadratic equation which can easily be factorised

$$\therefore (2k-5)(k+4)=0$$

$$\therefore 2k-5=0 \quad \text{or} \quad k+4=0$$

$$k=\frac{5}{2} \quad \text{or} \quad k=-4$$

Step 3: substitute the original variables back in

$$\therefore 2x-7=\frac{5}{2} \quad \text{or} \quad 2x-7=-4$$

$$\therefore x=\frac{19}{4} \quad \text{or} \quad x=\frac{3}{2}$$

Solve for x if $2x^4-10x^2-28=0$

Step 1: substitute $k=x^2$ to transform the complicated equation to

$$2k^2-10k-28=0$$

$$\therefore k^2-5k-14=0$$

Step 2: this is a quadratic equation which can easily be factorised

$$\therefore (k-7)(k+2)=0$$

$$\therefore k-7=0 \quad \text{or} \quad k+2=0$$

$$k=7 \quad \text{or} \quad k=-2$$

Step 3: substitute the original variables back in

$$x^2=7 \quad \text{or} \quad x^2=-2$$

$$\therefore x=\pm\sqrt{7} \quad \text{or} \quad x=\pm\sqrt{-2}$$ ← This is impossible

$$\therefore x=\sqrt{7} \quad \text{or} \quad x=-\sqrt{7}$$

1. Substitute $k = x^3$ to solve for x in $2x^6 + 4x^3 - 16 = 0$.

2. Make a substitution to solve for x in:

a) $40 + 6(7x + 3) - (7x + 3)^2 = 0$

b) $(x - 2)^4 - 5(x - 2)^2 - 36 = 0$

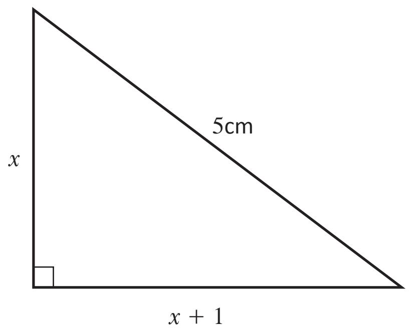
c) $9^x - 10(3^x) + 9 = 0$
(Hint: $9 = 3^2$)

Quadratic Word Problems

Some problems can be represented by a quadratic equation. Any of the methods in this chapter can be used to solve these problems.

When choosing a method to use, most people find it easier to try factorising into two brackets first. If an equation can't be factorised into two brackets easily, then use the quadratic formula or completing the square.

Use the Pythagorean Theorem to find the lengths of the missing sides in the following right angled triangle:



According to the Pythagorean theorem: $x^2 + (x+1)^2 = 5^2$

$\therefore 2x^2 + 2x - 24 = 0$

$\therefore x^2 + x - 12 = 0$

This can be factorised into two brackets $(x-3)(x+4) = 0$

$\therefore x = 3$ or $x = -4$

The length of the side of a triangle can't be negative, so $x = 3$ is the only solution.
The missing sides of the triangle are $x = 3$cm and $x + 1 = 4$cm

Here is an example of a word problem which can be represented by a quadratic equation. To find an equation from a word problem, let a variable equal the missing value and use the information in the word problem to create an equation.

Sandra is twice as old as Russell. Eight years ago, the product of their ages was 10. Find their current ages:

Let Russell's age be x

$\therefore$ Sandra's age is $2x$

Eight years ago their ages were: Russell: $x - 8$
Sandra: $2x - 8$

From the word problem we can say $(2x-8)(x-8) = 10$

$\therefore 2x^2 - 24x + 64 = 10$

$2x^2 - 24x + 54 = 0$

$\therefore x^2 - 12x + 27 = 0$

This can be factorised into two brackets $(x-3)(x-9) = 0$

$\therefore x = 3$ or $x = 9$

It is impossible for Russell to be 3 years old since he would not yet have been born 8 years ago.

$\therefore x = 9$ is the only solution

$\therefore$ Russell is currently 9 years and Sandra is currently $2(9) = 18$ years old.

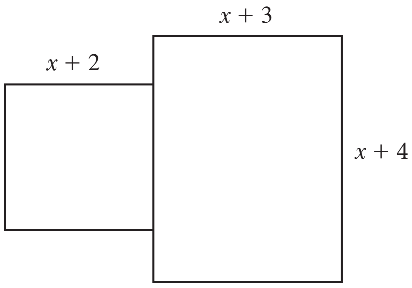
1. The product of two *consecutive* integers is 272.

a Find the two numbers if they are positive.

b Find the two numbers if they are negative.

2. A rectangle's length is 3 less than four times the breadth. Find the dimensions of the rectangle if the area is 126cm^2 :

3. The stage below is made up of a square and a rectangle. Find x If the total area of the stage is 191m^2.



Simultaneous Equations with Quadratics

In the same way simultaneous equations exist for linear equations, they exist for quadratic equations too. When you are solving simultaneous equations involving a linear equation and a quadratic equation always follow these four steps:

***Step 1*:** Make y the subject in both equations

***Step 2*:** Substitute the quadratic equation into the linear equation

***Step 3*:** Solve for x

***Step 4*:** Substitute the x-value into the original linear equation (or the original quadratic equation) to solve for y

Solve for these simultaneous equations

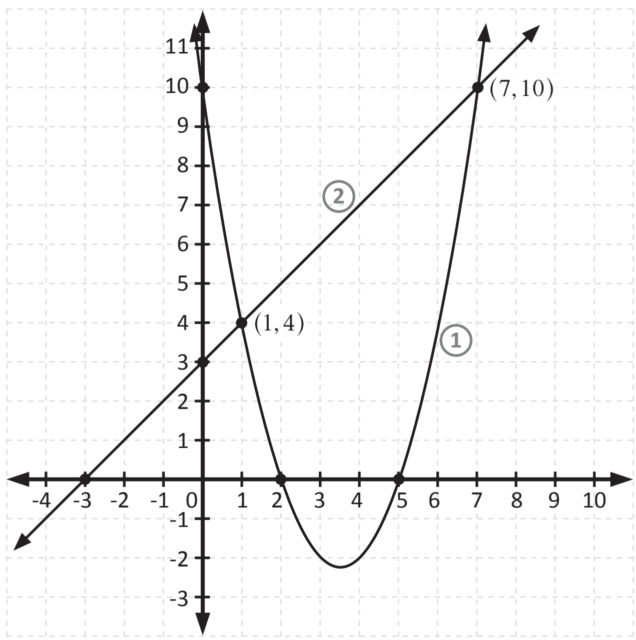
$y - 10 = x^2 - 7x$

$y - x = 3$

***Step 1*:** Make y the subject of both equations

$y = x^2 - 7x + 10$ ①

$y = x + 3$ ②



***Step 2*:** Substitute the quadratic equation into the linear equation

Substitute ② into ①

$(x^2 - 7x + 10) = x + 3$

***Step 3*:** Solve for x

$$x^2 - 7x + 10 - x - 3 = 0$$

$$x^2 - 8x + 7 = 0$$

$$(x - 1)(x - 7) = 0$$

$\therefore x = 1$ or $x = 7$

***Step 4*:** Substitute the x-value into the original linear equation (or the original quadratic equation) to solve for y

$y = (1) + 3$ or $y = (7) + 3$

$y = 4$ $\quad$ $y = 10$

$\therefore x = 1$ and $y = 4$ or $x = 7$ and $y = 10$

This means that the graphs of the equations will intersect at $(1, 4)$ and $(7, 10)$

A straight line (linear equation) and parabola (quadratic equation) could intersect at one point, two points or not at all.

- If the quadratic equation in *step 3* has two solutions, then the straight line and parabola intersect at two points and so the straight line is a secant.
- If the equation in *step 3* has one solution (the solutions are equal) then the straight line touches the parabola only once and so the straight line is a tangent to the parabola.
- If the equation in *step 3* has no real solutions $(b^2 - ac < 0)$ then the straight line and parabola will not intersect at all.

4. Solve these simultaneous equations for x and y. How many times does the straight line intersect the parabola?

$y = x^2 + 13x + 1$

$y = 8x + 1$

5. Solve these simultaneous equations for x and y. How many times does the straight line intersect the parabola?

$y - 5 = 4x^2 - 10x$

$y + 3 = 2x$

6. Solve these simultaneous equations for x and y. Find the coordinates of the point(s) where the graphs would intersect:

$y - 3x = x^2 - 37$

$-23 = 7x - y$

7. Solve these simultaneous equations for x and y. Find the coordinates of the point(s) where the graphs would intersect:

$y = 3x^2 - 9x + 30$

$y = -2x + 15$

Basics:

1. In a linear equation the highest index of the variable (eg x) has an index of 1. In a quadratic equation the highest index of the variable has an index of 2 (eg x^2).

2.
a) $x = \pm 2$
Two solutions

b) $b = \pm 4$
Two solutions

c) $b = \pm 5$
Two solutions

d) $y = 0$
One solution

e) $t = \pm\sqrt{7}$
Two solutions

f) $h^2 = -9$
No real solutions

g) $p^2 = -4$
No real solutions

h) $m = 0$
One solution

i) $x = \pm\frac{4}{3}$
Two solutions

j) $k = \pm 6$
Two solutions

Knowing More:

1.
a) $x^2 + 3x + 2 = 0$
$\therefore a = 1,\ b = 3,\ c = 2$

b) $2x^2 + 4x + 5 = 0$
$\therefore a = 2,\ b = 4,\ c = 5$

c) $x^2 + 0x - 7 = 0$
$\therefore a = 1,\ b = 0,\ c = -7$

d) $x^2 + 4x + 0 = 0$
$\therefore a = 1,\ b = 4,\ c = 0$

e) $12x^2 - 15x + 0 = 0$
$\therefore a = 12,\ b = -15,\ c = 0$

f) $x^2 - 4x - 21 = 0$
$\therefore a = 1,\ b = -4,\ c = -21$

g) $3x^2 - 19x - 40 = 0$
$\therefore a = 3,\ b = -19,\ c = -40$

h) $-3x^2 - 9x + 12 = 0$
$\therefore a = -3,\ b = -9,\ c = 12$

Knowing More:

2.
a) $x = 0$ or $x - 2 = 0$ (Null Factor Law)
$x = 0$ or $x = 2$

b) $x = 0$ or $x + 4 = 0$ (Null Factor Law)
$x = 0$ or $x = -4$

c) $x = 0$ or $2x + 1 = 0$ (Null Factor Law)
$x = 0$ or $x = -\frac{1}{2}$

d) $x - 3 = 0$ or $x + 5 = 0$ (Null Factor Law)
$x = 3$ or $x = -5$

e) $y = 0$ or $2y - 3 = 0$ (Null Factor Law)
$y = 0$ or $y = \frac{3}{2}$

f) $2x + 7 = 0$ or $3x - 8 = 0$ (Null Factor Law)
$x = -\frac{7}{2}$ or $x = \frac{8}{3}$

3.
a) $x = 0$ or $x + 8 = 0$ (Null Factor Law)
$x = 0$ or $x = -8$

b) $4x = 0$ or $x + 3 = 0$ (Null Factor Law)
$x = 0$ or $x = -3$

c) $t = 0$ or $t - 6 = 0$ (Null Factor Law)
$t = 0$ or $t = 6$

d) $5b = 0$ or $b - 3 = 0$ (Null Factor Law)
$b = 0$ or $b = 3$

e) $y - 7 = 0$ or $y + 3 = 0$ (Null Factor Law)
$y = 7$ or $y = -3$

f) $m - 6 = 0$ or $m + 5 = 0$ (Null Factor Law)
$m = 6$ or $m = -5$

Knowing More:

3. g $n + 8 = 0$ or $n - 4 = 0$ (Null Factor Law)
$n = -8$ or $n = 4$

h $x - 1 = 0$ or $x + 6 = 0$ (Null Factor Law)
$x = 1$ or $x = -6$

i $x + 7 = 0$ or $x - 5 = 0$ (Null Factor Law)
$x = -7$ or $x = 5$

j $p - 9 = 0$ or $p + 7 = 0$ (Null Factor Law)
$p = 9$ or $p = -7$

4. a $(x+2)^2 - 3$ b $(x-3)^2 + 7$

c $\left(x + \frac{3}{2}\right)^2 + \frac{19}{4}$ d $3(x+4)^2 - 18$

e $-(x-5)^2 + 2$ f $-3(x+1)^2 + 21$

5. a $x = -2 + \sqrt{5}$ or $x = -2 - \sqrt{5}$

b $x = -1$ or $x = 7$

c $x = 7 + 2\sqrt{\frac{2}{3}}$ or $x = 7 - 2\sqrt{\frac{2}{3}}$

d No real solution

6. a $x = -3 + \sqrt{7}$ or $x = -3 - \sqrt{7}$

b $x = 5 + \sqrt{10}$ or $x = 5 - \sqrt{10}$

7. a $q = \frac{1}{2}$ or $q = -2$

b $m = -1 + \sqrt{6}$ or $m = -1 - \sqrt{6}$

c $t = -1 + \sqrt{\frac{5}{4}}$ or $t = -1 - \sqrt{\frac{5}{4}}$

d $x = 2 + \sqrt{\frac{10}{3}}$ or $x = 2 - \sqrt{\frac{10}{3}}$

Knowing More:

8. a $\left(x + \frac{b}{2a}\right)^2 = \frac{b^2}{4a^2} - \frac{c}{a}$

b $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

Using Our Knowledge:

1. a $x = 6$ or $x = -3$

b $x = 7$ or $x = 2$

c $x = -8$ or $x = -5$

d $x = 5$ or $x = 3$

e $x = -1 + \sqrt{6}$ or $x = -1 - \sqrt{6}$

f $x = -\frac{5}{2} + \frac{3\sqrt{5}}{2}$ or $x = -\frac{5}{2} - \frac{3\sqrt{5}}{2}$

2. a $x = -4$ or $x = \frac{2}{3}$

b $x = \frac{3}{2}$ or $x = -\frac{3}{2}$

c $x = \frac{7 + \sqrt{17}}{4}$ or $x = \frac{7 - \sqrt{17}}{4}$

d $x = \frac{-4 + 2\sqrt{34}}{-10}$ or $x = \frac{-4 - 2\sqrt{34}}{-10}$

3. a $7[(x-2)(x+8)] = 0$
$x - 2 = 0$ or $x + 8 = 0$ (Null Factor Law)
$x = 2$ or $x = -8$

b $(x+3)^2 = (\pm 5)^2$
$x = 2$ or $x = -8$

c $x = \frac{-42 \pm \sqrt{70}}{14}$
$x = -8$ or $x = 2$

Using Our Knowledge:

4. a $\Delta > 0$ and is a perfect square
$\therefore$ the solutions will be unequal and rational

b $\Delta > 0$ and is a perfect square
$\therefore$ the solutions will be unequal and rational

c $\Delta < 0$
$\therefore$ there are no real solutions for x

d $\Delta > 0$ and is a perfect square
$\therefore$ the solutions will be unequal and rational

e $\Delta > 0$ and is not a perfect square
$\therefore$ the solutions will be unequal and irrational

f $\Delta = 0$. The two solutions will be equal
These equal solutions will also be rational

5. a $rx^2 - 3x + 9 = 0$ has one solution when $r = \frac{1}{4}$

b The equation $rx^2 - 3x + 9 = 0$ has unequal solutions when $r < \frac{1}{4}$

6. $x^2 + x + 7k = 6k - 3x$ has no real solutions when $k > 4$

9. $\frac{2}{p} = \frac{2 - 2x^2}{5 - 4x}$ has two unequal and real solutions when $p < 1$ or $p > 4$.

Thinking More:

1. $\therefore x = \sqrt[3]{2}$ or $x = \sqrt[3]{-4}$

2. a $\therefore x = -1$ or $x = 1$

b $\therefore x = -1$ or $x = 5$

c $\therefore x = 2$ or $x = 0$

Thinking Even More:

1. a The positive numbers are 16 and 17

b The negative numbers are -16 and -17

2. Breadth $= 6$cm
Length is $= 21$cm

3. $x = 7$m

4. $x = 0$ or $x = -5$

$y = 1$ or $y = -39$

Therefore the straight line intersects the parabola twice at: $(0, 1)$ and $(-5, -39)$

5. $x = 2$ or $x = 1$

$y = 1$ or $y = -1$

Therefore the straight line intersects the parabola twice at: $(2, 1)$ and $(1, -1)$

6. $x = 10$ or $x = -6$

$y = 93$ or $y = -19$

Therefore the graphs intersect at: $(10, 93)$ and $(-6, -19)$

7. The discriminant is < 0 so this can't be solved (no real solutions)